Obstructions to deforming curves on a prime Fano threefold

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Given a projective scheme V, we denote by $\mathrm{Hilb}^{sc} V$ the Hilbert scheme of smooth connected curves in V. Mumford [3] first proved that if $V = \mathbb{P}^3$, then there exists an irreducible component W of $\mathrm{Hilb}^{sc} V$, along which $\mathrm{Hilb}^{sc} V$ is nowhere reduced. Here we say that W is a *generically non-reduced component* of $\mathrm{Hilb}^{sc} V$. Later, this example has been generalized in the case $V = \mathbb{P}^3$, by many algebraic geometers, e.g., Kleppe, Ellia, Gruson-Peskine, Fløystad, and Kleppe-Ottem[1], etc. Recently, it has been generalized in [2, 4] also for many uniruled 3-folds V, e.g., a smooth Fano 3-fold of index at least 2.

In this talk, I will discuss the existence of a generically non-reduced component of $\mathrm{Hilb}^{sc} V$ for every prime Fano 3-fold V, i.e, every smooth Fano 3-fold V of index 1 with $\mathrm{Pic}\, V \simeq \mathbb{Z}$. We consider the deformations of smooth curves C on a smooth Fano 3-fold V, assuming that C is contained in a smooth anti-canonical member $S \in |-K_V|$ of V, i.e., a $K3$ surface S in V. We give a sufficient condition for C to be obstructed in V, in terms of (-2)-curves and elliptic curves on S. We apply this result and prove that for every prime Fano 3-fold V of genus g $(= (-K_V)^3/2 + 1)$, $\mathrm{Hilb}^{sc} V$ contains a generically non-reduced component of dimension $5g + 1$, which becomes a variation of Mumford's example for $\mathrm{Hilb}^{sc} \mathbb{P}^3$. This talk is based on the results in [5, 6].

References

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