The Hilbert scheme of canonical curves in del Pezzo 3-folds and its application to the Hom scheme. *

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Abstract: Modifying Mumford's example, we construct a generically non-reduced component of the Hilbert scheme $\mathrm{Hilb}^S V_d$ parametrizing smooth connected curves in a smooth del Pezzo 3-fold $V_d \subset \mathbb{P}^{d+1}$ of degree d. As its application, we construct a new example of a generically non-reduced component of the Grothendieck's Hom scheme $\mathrm{Hom}(X, V_3)$ parametrizing morphisms from a general curve X of genus 5 to a general cubic 3-fold V_3.

§0 Introduction

For given projective scheme V, $\mathrm{Hilb}^S V$ denotes the Hilbert scheme of smooth connected curves in V. Mumford[13] showed that the Hilbert scheme $\mathrm{Hilb}^S \mathbb{P}^3$ contains a generically non-reduced (irreducible) component. Let $V_d \subset \mathbb{P}^{d+1}$ be a smooth del Pezzo 3-fold of degree d. In this article, modifying and simplifying Mumford's example, we construct a generically non-reduced component of $\mathrm{Hilb}^S V_d$ as an example of the Hilbert scheme of curves in other Fano 3-folds.

Theorem 1. Let $V_d \subset \mathbb{P}^{d+1}$ be a smooth del Pezzo 3-fold of degree d. Then $\mathrm{Hilb}^S V_d$ has an irreducible component W which is generically non-reduced.

Every canonical curve of genus $g = d+2$ is contained in the projective space $\mathbb{P}^{d+1}$. We consider the irreducible components of $\mathrm{Hilb}^S V_d$ whose general member is an embedding of a canonical curve X into $V_d \subset \mathbb{P}^{d+1}$. There are two kinds of embeddings $f : X \hookrightarrow V_d$: one is linearly normal (i.e. $H^1(\mathcal{I}_{f(X)/V_d}(1)) = 0$) and the other is linearly non-normal (i.e. $H^1(\mathcal{I}_{f(X)/V_d}(1)) \neq 0$). Correspondingly, there exists (at least) two irreducible components of $\mathrm{Hilb}^S V_d$. One is generically reduced and the other is generically non-reduced. A general

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member of the generically reduced one is linearly normal, while that of the generically non-reduced one is linearly non-normal. The irreducible component W of the theorem is the second one.

A general member of W is a curve $C \subset V_d$ contained in a smooth hyperplane section $S_d = V_d \cap H$, that is a del Pezzo surface of degree d, and the linear span $\langle C \rangle \subset \mathbb{P}^{d+1}$ is $\mathbb{P}^d$. Moreover we see that C is a projection of a canonical curve $X \subset \mathbb{P}^{d+1}$ from a general point of $\mathbb{P}^{d+1}$.

$$\begin{array}{rccccccc} & \mathbb{P}^{d+1} & \supset & X & & V_d & \subset & \mathbb{P}^{d+1} \\ \text{generic projection} & \downarrow & & \downarrow \simeq & & \cup & & \uparrow \\ & \mathbb{P}^d & \supset & C & \hookrightarrow & S_d = V_d \cap H & \subset & H \simeq \mathbb{P}^d \end{array}$$

We apply Theorem 1 for $d = 3$ (i.e. V_d is a cubic 3-fold $V_3 \subset \mathbb{P}^4$) to show the non-reducedness of the Hom scheme.

For given two projective schemes X and V, the set of morphisms $f : X \to V$ has a natural scheme structure as a subscheme of the Hilbert scheme of $X \times V$. We call this scheme the *Hom scheme* and denote by $\operatorname{Hom}(X, V)$. When we fix a projective embedding $V \hookrightarrow \mathbb{P}^n$, or a polarization $\mathcal{O}_V(1)$ of V more generally, all the morphisms of degree d are parametrized by an open and closed subscheme, which we denote by $\operatorname{Hom}_d(X, V)$.

In what follows, we assume that both X and V are smooth and X is a curve. It is well known that the Zariski tangent space of $\operatorname{Hom}(X, V)$ at $[f]$ is isomorphic to $H^0(X, f^*\mathcal{T}_V)$ and the following dimension estimate holds:

$$\deg f^*(-K_V) + n(1-g) \leq \dim_{[f]} \operatorname{Hom}(X, V) \leq \dim H^0(X, f^*\mathcal{T}_V), \tag{1}$$

where $n = \dim V$, g is the genus of X and $\mathcal{T}_V$ is the tangent bundle of V. The lower bound is equal to $\chi(f^*\mathcal{T}_V)$ and called the *expected dimension.*

The Hom scheme from a curve plays a central role in Mori theory and the study of Gromov-Witten invariants. However we do not have many examples of the Hom scheme, especially of those from irrational curves. In this article we study the Hom scheme $\operatorname{Hom}_8(X, V_3)$ of morphisms of degree 8 from a general curve X of genus 5 to a smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$ and show the following:

Theorem 2. Assume that V_3 is either general or of Fermat type

$$V_3^{\text{Fermat}} : x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \quad \subset \mathbb{P}^4.$$

Then $\operatorname{Hom}_8(X, V_3)$ has an irreducible component T of expected dimension $(= 4)$ which is generically non-reduced.

Remark 3. (1) The expected dimension is equal to $2d+3(1-g)=4$ since $\mathcal{O}_V(-K_{V_3})\simeq\mathcal{O}_{V_3}(2)$. The tangential dimension of $\mathrm{Hom}_8(X,V_3)$ at a general point $[f]\in T$ is equal to $h^0(f^*\mathcal{T}_{V_3})=5$.

(2) It is known that the Hom schemes $\mathrm{Hom}_1(\mathbb{P}^1,V)$ from $\mathbb{P}^1$ to certain special Fano 3-folds V are generically non-reduced (cf. §3.3).

Mumford constructed the generically non-reduced component of $\mathrm{Hilb}^S\,\mathbb{P}^3$ to show the *pathology* of the Hilbert schemes. After his study, by the many continued works [7], [9], [5], [4], [6] and [10], we have seen that non-reduced components frequently appear in $\mathrm{Hilb}^S\,\mathbb{P}^3$. Thus the non-reducedness itself is no longer pathology now. However the non-reducedness seems to be derived from case by case reasons. One of the motivation of our work is to find more intrinsic reason for the non-reducedness of the Hilbert schemes and the Hom schemes (if there exists).

We proceed in this article as follows. In §1 we prove Theorem 1. As a special case of the theorem, we show that the Hilbert scheme $\mathrm{Hilb}^S\,V_3$ has a generically non-reduced component $\tilde{W}$. In §2 we consider a natural morphism $\varphi:\tilde{W}\to\mathfrak{M}_5$ (classification morphism) from $\tilde{W}$ to the moduli space $\mathfrak{M}_5$ of curves of genus 5 and prove its dominance. Since a general fiber of φ is birationally equivalent to a component T of the Hom scheme $\mathrm{Hom}(X,V_3)$, we deduce Theorem 2 from the smoothness of $\mathfrak{M}_5$. Finally we see other examples concerning non-reduced components of the Hilbert schemes and Hom schemes in §3. We work over an algebraically closed field k of characteristic 0 throughout.

Notation 4. For a given algebraic variety V, $\mathrm{Hilb}^S_{d,g}\,V$ denotes the subscheme of $\mathrm{Hilb}^S\,V$ consisting of curves of degree d and genus g. $\mathrm{Hilb}^S\,V$ is the disjoint union $\bigsqcup_{(d,g)\in\mathbb{Z}^2}\mathrm{Hilb}^S_{d,g}\,V$.

§1 Non-reduced components of the Hilbert scheme

In this section, we show that for every smooth del Pezzo 3-fold $V_d\subset\mathbb{P}^{d+1}$, the Hilbert scheme $\mathrm{Hilb}^S\,V_d$ has a generically non-reduced component of dimension $4d+4$.

Del Pezzo 3-folds A smooth 3-fold $V_d\subset\mathbb{P}^{d+1}$ is called *del Pezzo* (of degree d) if every linear section $[V_d\subset\mathbb{P}^{d+1}]\cap H_1\cap H_2$ with general two hyperplanes $H_1,H_2\subset\mathbb{P}^{d+1}$ is an elliptic normal curve $F_d\subset\mathbb{P}^{d-1}$ (of degree d).

Example 5. [del Pezzo 3-folds]

del Pezzo 3-folds	degree	
$V_3 = (3) \subset \mathbb{P}^4$	3	cubic hypersurface
$V_4 = (2) \cap (2) \subset \mathbb{P}^5$	4	complete intersection
$V_5 = [\mathrm{Gr}(2,5) \overset{\text{Plücker}}{\hookrightarrow} \mathbb{P}^9] \cap H_1 \cap H_2 \cap H_3$	5	linear section of Grassmannian
$V_6 = [\mathbb{P}^2 \times \mathbb{P}^2 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^8] \cap H$	6	
$V_6' = [\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^7]$	6	
$V_7 = \mathrm{Blow}_{\mathrm{pt}} \mathbb{P}^3 \subset \mathbb{P}^8$	7	blow-up of $\mathbb{P}^3$ at a point
$V_8 = \mathbb{P}^3 \overset{\text{Veronese}}{\hookrightarrow} \mathbb{P}^9$	8	

Remark 6. The del Pezzo 3-folds V_d of degree $d = 1$ and $d = 2$ are also known. They can be realized as a hypersurface of a weighted projective space.

Let $V_d \subset \mathbb{P}^{d+1}$ be a smooth del Pezzo 3-fold of degree $d \leq 7$, and let $S_d = V_d \cap H$ be a smooth hyperplane section of V_d, and let E be a line contained in S_d. All pairs (S_d, E) of such S_d and E are parametrized by an open subset P of a $\mathbb{P}^{d-1}$-bundle over the Fano surface $F \subset G(1, \mathbb{P}^{d+1})$ of lines on V_d. We consider the complete linear system $\Lambda := |-2K_{S_d} + 2E|$ on S_d. Then Λ is the pull-back of $|-2K_{S_{d+1}}| \simeq \mathbb{P}^{3(d+1)}$ on the surface S_{d+1}, the blow-down of E on S_d. Λ is base point free and every general member C of Λ is a smooth connected curve of degree $2d+2$ and genus $d+2$. All such curves C are parametrised by an open subset W of a $\mathbb{P}^{3d+3}$-bundle over P. Thus we have a diagram

$$\begin{array}{rcl}
\{(S_d, C) | C \in |-2K_{S_d} + 2E|\} & = & W \ \longrightarrow \ \mathrm{Hilb}^S_{2d+2,d+2} V_d \\
 & & \Big\downarrow {\scriptstyle \mathbb{P}^{3d+3}\text{-bundle}} \\
\{(S_d, E) | E \subset S_d\} & = & P \\
 & & \Big\downarrow {\scriptstyle \mathbb{P}^{d-1}\text{-bundle}} \\
\{E \subset V_d\} & = & F.
\end{array}$$

Since $\deg C = 2d + 2 > d = \deg V_d$, C is contained in a unique hyperplane section S_d. Moreover, $E \subset S_d$ is recovered from C as the unique member of $|\frac{1}{2}C + K_{S_d}|$. Therefore the classification morphism $W \to \mathrm{Hilb}^S V_d$ is an embedding. In particular, the Kodaira-Spencer map

$$\kappa_{[C]} : t_{W,[C]} \longrightarrow H^0(C, N_{C/V_d}) \tag{2}$$

of the family W is injective at any point $[C] \in W$. In what follows, we regard W as a subscheme of $\mathrm{Hilb}^S V_d$. Let us consider the exact sequence of normal bundles

$$0 \longrightarrow \underbrace{N_{C/S_d}}_{\cong \mathcal{O}_C(2K_C)} \longrightarrow N_{C/V_d} \longrightarrow \underbrace{N_{S_d/V_d}\big|_C}_{\cong \mathcal{O}_C(K_C)} \longrightarrow 0. \tag{3}$$

Note that the dimension of the tangent space $H^0(C, N_{C/V_d})$ of $\operatorname{Hilb}^S V$ at $[C]$ is equal to

$$\begin{aligned} h^0(N_{C/V_d}) &= h^0(2K_C) + h^0(K_C) \\ &= (3d+3) + (d+2) \\ &= 4d+5 \\ &> \dim W = 4d+4. \end{aligned}$$

Therefore there exists the following two possibilities:

(A) The Zariski closure $\overline{W}$ of W is an irreducible component of $(\operatorname{Hilb}^S V_d)_{\mathrm{red}}$ and $\operatorname{Hilb}^S V_d$ is singular along W;

(B) There exists an irreducible component Z of $\operatorname{Hilb}^S V_d$ such that $Z \supsetneq W$ and $\operatorname{Hilb}^S V_d$ is generically smooth along W.

The case (A) automatically implies that $\operatorname{Hilb}^S V_d$ is generically non-reduced along W since W is a component. We prove that the case (B) does not occur.

Theorem 7. The Zariski closure $\overline{W}$ of W is an irreducible component of $(\operatorname{Hilb}^S_{2d+2,d+2} V_d)_{\mathrm{red}}$ of dimension $4d+4$, and $\operatorname{Hilb}^S V_d$ is generically non-reduced along W.

For the proof, we use infinitesimal analysis of the Hilbert scheme (infinitesimal deformations and their obstructions) which was used in [14],[2]. (In the case $d = 3$, there is another approach, which is similar to the method used by Mumford in [13].

Infinitesimal analysis of the Hilbert scheme Let C be a curve on an algebraic variety V. An *(embedded) first order infinitesimal deformation* of $C \hookrightarrow V$ is a closed subscheme $\tilde{C} \subset V \times \operatorname{Spec} k[t]/(t^2)$ which is flat over $\operatorname{Spec} k[t]/(t^2)$ and $\tilde{C} \times k = C$. The set of all first order deformations of $C \hookrightarrow V$ are parametrized by $H^0(N_{C/V})$ and isomorphic to the tangent space of the Hilbert scheme $\operatorname{Hilb}^S V$ at the point $[C]$. If $\operatorname{Hilb}^S V$ is smooth at $[C]$, then for every $\alpha \in H^0(N_{C/V})$ and every integer $n \geq 3$, the corresponding infinitesimal first order deformation C_α of $C \hookrightarrow V$ lifts to a deformation over $\operatorname{Spec} k[t]/(t^n)$.

Proposition 8. Let C be a smooth connected curve on a smooth del Pezzo 3-fold V_d of degree $d \leq 7$. Assume that C is contained in a smooth hyperplane section S_d of V_d and $C \sim -2K_{S_d} + 2E$ for a line E on S_d. If N_{E/V_d} is trivial, then for any $\alpha \in H^0(C, N_{C/V_d}) \setminus \operatorname{im} \kappa_{[C]}$ (cf. (2)) the first order infinitesimal deformation C_α of C does not lift to a deformation over $\operatorname{Spec} k[t]/(t^3)$. (i.e. the obstruction $\operatorname{ob}(\alpha)$ is nonzero.)

Fact 9 (Iskovskih). Let E be a line on a smooth del Pezzo 3-fold V_d of degree $d \leq 7$ and let N_{E/V_d} be the normal bundle. Then there are only the following possibilities:

$$\begin{array}{lll} (0,0): & N_{E/V_d} \simeq \mathcal{O}_{\mathbb{P}^1}{}^{\oplus 2}, & \cdots \quad (\textit{good line}) \\ (1,\text{-}1): & N_{E/V_d} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1), & \\ (2,\text{-}2): & N_{E/V_d} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2), \quad (\text{only if } d = 1 \text{ or } 2) & \Bigg\} \quad (\textit{bad line}) \\ (3,\text{-}3): & N_{E/V_d} \simeq \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(3). \quad (\text{only if } d = 1) & \end{array}$$

Every general line E is a good line. All bad lines are parametrized by a curve on the Fano surface F of lines on V_d.

Theorem 7 follows from Proposition 8 and Fact 9 in the following way.

Proof of Theorem 7 Let C be a general member of the irreducible closed subset W. We have the natural inequalities

$$\dim W \leq \dim_{[C]} \operatorname{Hilb}^S V_d \leq h^0(C, N_{C/V_d}). \tag{4}$$

Since C is general, it follows from Fact 9 that $E := 1/2(C+2K_S)$ is a good line. Therefore by Proposition 8, $C \hookrightarrow V_d$ has a first order infinitesimal deformation that does not lift to a deformation over $\operatorname{Spec} k[t]/(t^3)$. Hence we have $\dim_{[C]} \operatorname{Hilb}^S V_d < h^0(C, N_{C/V_d})$. Note that $h^0(C, N_{C/V_d}) - \dim W = 1$. This indicates $\dim W = \dim_{[C]} \operatorname{Hilb}^S V_d$. In particular, W is an irreducible component of $(\operatorname{Hilb}^S V_d)_{\mathrm{red}}$. Since $\operatorname{Hilb}^S V_d$ is singular at every general point $[C]$ of W, $\operatorname{Hilb}^S V_d$ is non-reduced along W. □

Since V_8 is isomorphic to $\mathbb{P}^3$, $\operatorname{Hilb}^S V_8$ has a generically non-reduced component (cf. [13]). Thus we obtain Theorem 1 from Theorem 7.

We prove Proposition 8 by a criterion using cup products on cohomology groups. More precisely, we show that the obstruction $\operatorname{ob}(\alpha)$ is nonzero for every $\alpha \in H^0(N_{C/V_d}) \setminus \operatorname{im} \kappa_{[C]}$.

Lemma 10. Let C be a smooth connected curve on a smooth variety V and let $\alpha \in H^0(N_{C/V}) \simeq \operatorname{Hom}(\mathcal{I}_{C/V}, \mathcal{O}_C)$ be a global section of the normal bundle $N_{C/V}$. Then the first order infinitesimal deformation $\tilde{C} \subset V \times \operatorname{Spec} k[t]/(t^2)$ corresponding to α lifts to a deformation over $\operatorname{Spec} k[t]/(t^3)$ if and only if the cup product

$$\operatorname{ob}(\alpha) := \alpha \cup \mathbf{e} \cup \alpha \in \operatorname{Ext}^1(\mathcal{I}_{C/V}, \mathcal{O}_C).$$

is zero, where $\mathbf{e} \in \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{I}_{C/V})$ is the extension class of the natural exact sequence $0 \to \mathcal{I}_{C/V} \to \mathcal{O}_V \to \mathcal{O}_C \to 0$.

We cut the computation of $\operatorname{ob}(\alpha)$ and the proof of its nonzero in this article.

Above non-reduced component of $\mathrm{Hilb}^S V_d$ can be generalized as follows. In its construction, we considered a family W of curves $C \subset V_d$ lying on a smooth del Pezzo surface $S_d = H \cap V_d$. Every member C of W has an extra first order infinitesimal deformation of $C \hookrightarrow V_d$ other than the ones coming from W (i.e. $\dim W < H^0(N_{C/V_d})$). By a systematic study of the families W of such curves C, we obtain the next theorem. In what follows, we assume $d = 3$ (i.e. V_d is a smooth cubic 3-fold V_3) for simplicity.

Theorem 11. Let $e > 5$ and $g \geq e - 3$ be two integers, and let $W \subset \mathrm{Hilb}^S_{e,g} V_3$ be an irreducible closed subset whose general member C is contained in a smooth hyperplane section of V_d. Assume that W is maximal among all such subsets. Then we have the following:

(1) If $\rho := \dim H^1(V_3, \mathcal{I}_{C/V_3}(1)) = 0$ or 1, then W is an irreducible component of $(\mathrm{Hilb}^S V_3)_{\mathrm{red}}$ of dimension $e + g + 3$;

(2) $\mathrm{Hilb}^S V_3$ is generically smooth along W if $\rho = 0$, and is generically non-reduced along W if $\rho = 1$.

We give an example which is an application of Theorem 11. It is well known that a smooth cubic surface $S_3 \subset \mathbb{P}^3$ is isomorphic to a blown-up of $\mathbb{P}^2$ at 6-points. For each curve C on S_3, we have a 7-tuple $(a; b_1, \ldots, b_6)$ of integers as the divisor class $[C] \in \mathrm{Pic}\, S_3 \simeq \mathbb{Z}^7$. The 7-tuple is uniquely determined from C up to the symmetry with respect to the action $W(\mathbb{E}_6) \curvearrowright \mathrm{Pic}\, S_3$ of the Weyl group $W(\mathbb{E}_6)$.

Definition 12. Let V_3 be a smooth cubic 3-fold. For a given 7-tuple $(a; b_1, \ldots, b_6)$ of integers, we define an irreducible closed subset $W_{(a;b_1,\ldots,b_6)} \subset \mathrm{Hilb}^S V_3$ whose general member C is contained in a smooth hyperplane section (i.e. smooth cubic surface) S_3 of V_3 by

$$W_{(a;b_1,\ldots,b_6)} := \left\{ C \in \mathrm{Hilb}^S V_3 \mid C \subset {}^{\exists}S_3 : \text{smooth cubic}, \quad C \in |\mathcal{O}_S(a : b_1, \ldots, b_6)| \right\}^- .$$

Here $^-$ denotes the Zariski closure in $\mathrm{Hilb}^S V_3$.

Example 13. Let $\lambda \in \mathbb{Z}_{\geq 0}$ and let W be one of the irreducible closed subsets

$$\begin{aligned} W =& W_{(\lambda+6;\lambda+1,1,1,1,1,0)} \subset \mathrm{Hilb}^S_{e,2e-16} V_3 \quad (e = 2\lambda + 13) \quad \text{or} \\ W =& W_{(\lambda+6;\lambda+2,1,1,1,1,0)} \subset \mathrm{Hilb}^S_{e,\frac{3}{2}e-9} V_3 \quad (e = 2\lambda + 12). \end{aligned}$$

Then a general member C of W satisfies $h^1(C, \mathcal{I}_{C/V_3}(1)) = 1$. Therefore by Theorem 11 W is an irreducible component of $(\mathrm{Hilb}^S V_3)_{\mathrm{red}}$ and $\mathrm{Hilb}^S V_3$ is generically non-reduced along W. In particular, $\mathrm{Hilb}^S V_3$ has infinitely many non-reduced components.

§2 Non-reduced components of the Hom scheme

In this section, we construct a new example of a generically non-reduced component of the Hom scheme. We will deduce the non-reducedness of the Hom scheme from that of the Hilbert scheme. By Theorem 7 in the case $d = 3$, we have shown that there exists a generically non-reduced component $\tilde{W}$ of the Hilbert scheme $\mathrm{Hilb}^S_{8,5} V_3$ (i.e. $(\tilde{W})_{\mathrm{red}} = \overline{W}$). Then there exists a natural morphism (called the *classification morphism*)

$$\varphi : \tilde{W} \longrightarrow \mathfrak{M}_5$$

from $\tilde{W}$ to the moduli space $\mathfrak{M}_5$ of curves of genus 5. Let X be a general curve of genus 5. The fiber $\varphi^{-1}([X])$ at the point $[X] \in \mathfrak{M}_5$ is isomorphic to an open subscheme of $\mathrm{Hom}(X, V_3)$. We show that its Zariski closure T in $\mathrm{Hom}(X, V_3)$ satisfies the requirement of Theorem 2. It is essential to prove that φ is dominant. For the proof of the dominance we use the next theorem of Sylvester.

Lemma 14 (Sylvester's pentahedoron theorem (cf. [3])). A general cubic form $F(y_0, y_1, y_2, y_3)$ of four variables is a sum $\sum_{i=0}^4 l_i(y_0, y_1, y_2, y_3)^3$ of the cubes of five linear forms l_i $(0 \le i \le 4)$.

Proof of Theorem 2 Let X be a general curve of genus 5. The canonical model of X, that is, the image of $X \overset{K_X}{\hookrightarrow} \mathbb{P}^4$, is a general complete intersection $q_1 = q_2 = q_3 = 0$ of three quadrics. Let q, q' be general members of the net of quadrics $< q_1, q_2, q_3 >$ and let S_4 be their complete intersection $q = q' = 0$. Then S_4 is a del Pezzo surface of degree 4. We denote the blow-up of S_4 at a general point $p \in S_4 \setminus X$ by $\pi_p : S_3 \to S_4$. Then we have a commutative diagram

$$\begin{array}{ccccc} X & \subset & S_4 & \subset & \mathbb{P}^4 \\ \| & & \uparrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \text{projection from } p} \\ C & \subset & S_3 & \subset & \mathbb{P}^3. \end{array} \tag{5}$$

Here C denote the inverse image of X by π_p. Since X belongs to the linear system $|-2K_{S_4}|$ on S_4, C belongs to $|\pi_p^*(-2K_{S_4})| = |-2K_{S_3} + 2E|$, where E is the exceptional curve of π_p. By the choice of q, q' and p, it follows that S_3 is a general cubic surface.

First we prove Theorem 2 in the case where V_3 is a cubic 3-fold of Fermat type V_3^{Fermat}. By Lemma 14, a general cubic surface is isomorphic to a hyperplane section of V_3^{Fermat}. Hence so is S_3. By the commutative diagram (5) the classification morphism $\varphi : \tilde{W} \to \mathfrak{M}_5$ is dominant, and general fiber T^{Fermat} is of dimension 4. Since $\mathfrak{M}_5$ is generically smooth, $\mathrm{Hom}(X, V_3^{\mathrm{Fermat}})$ is generically non-reduced along T^{Fermat}

Theorem 2 for a general V_3 follows from the Fermat case by the upper semi-continuity theorem on fiber dimensions. $\square$

Problem 15. Let $V_3 \subset \mathbb{P}^4$ be a cubic 3-fold and let $\mathfrak{M}_{\text{cubic}}$ be the moduli space of cubic surfaces. Is the classification map

$$\varphi_{V_3} : (\mathbb{P}^4)^* \dashrightarrow \mathfrak{M}_{\text{cubic}}, \qquad [H] \mapsto [H \cap V_3]$$

dominant for every smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$?

Remark 16. If we have the affirmative answer to the Problem 15, Theorem 2 is true for every smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$.

§3 Other examples

Let us see other examples concerning the non-reducedness of the Hilbert schemes and the Hom schemes.

§3.1 Curves on a Jacobian variety

A simple example of a generically non-reduced component of the Hilbert scheme is obtained from the Abel-Jacobi map $\alpha : C \hookrightarrow \operatorname{Jac} C$ of a curve C. Every deformation of α induces a deformation of $\operatorname{Jac} C \xrightarrow{\sim} \operatorname{Jac} C$. Therefore every deformation of $\alpha(C)$ as a subscheme of $\operatorname{Jac} C$ is a translation of $\alpha(C)$ in $\operatorname{Jac} C$ induced by the group structure of $\operatorname{Jac} C$. Hence $(\operatorname{Hilb}^S(\operatorname{Jac} C))_{\text{red}}$ contains an irreducible component $T \simeq \operatorname{Jac} C$ passing through $[\alpha(C)]$.

Proposition 17. If C is a hyperelliptic curve of genus $g \geq 3$, then the Hilbert scheme $\operatorname{Hilb}^S(\operatorname{Jac} C)$ is non-reduced along T.

Proof It suffices to show the non-reducedness at $[\alpha(C)]$. Let

$$\begin{array}{ccccc} 0 \to \mathcal{T}_C & \to & \mathcal{T}_{\operatorname{Jac} C}\big|_C & \to & N_{C/\operatorname{Jac} C} \to 0 \\ & & \| & & \\ & & H^1(\mathcal{O}_C) \otimes \mathcal{O}_C & & \end{array}$$

be the natural exact sequence. The induced linear map $H^1(\mathcal{T}_C) \to H^1(\mathcal{O}_C) \otimes H^1(\mathcal{O}_C)$ is not injective since $H^0(K_C) \otimes H^0(K_C) \to H^0(K_C^{\otimes 2})$ is not surjective by assumption and computation. Hence we have $\dim H^0(N_{C/\operatorname{Jac} C}) > g = \dim T$ by the exact sequence. $\square$

This non-reducedness is caused by the ramification of the period map $\mathfrak{M}_g \to \mathcal{A}_g$ along the hyperelliptic locus. The Hom scheme $\operatorname{Hom}(C, \operatorname{Jac} C)$ is non-singular at α.

§3.2 Mumford pathology

Mumford [13] proved that the Hilbert scheme $\mathrm{Hilb}^S_{14,24}\,\mathbb{P}^3$ of smooth connected curves in $\mathbb{P}^3$ of degree 14 and genus 24 has a generically non-reduced component W of expected dimension 56. A general member C of W is contained in a smooth cubic surface. It is linearly normal and not 3-normal (i.e. $H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0$). Since the dimension of the moduli space $\mathfrak{M}_{24}$ is bigger than $\dim W$, $[C] \in \mathrm{Hilb}^S_{14,24}\,\mathbb{P}^3$ is not general in $\mathfrak{M}_{24}$.

§3.3 Curves on Fano 3-folds

It is known that the Hilbert schemes $\mathrm{Hilb}_{1,0}\,V$ of lines on certain special Fano 3-folds V are generically non-reduced. Hence so are the Hom schemes $\mathrm{Hom}_1(\mathbb{P}^1, V)$ of morphisms of degree 1 with respect to $-K_V$. But in this case $\mathrm{Hilb}_{1,0}\,V'$ and hence $\mathrm{Hom}_1(\mathbb{P}^1, V')$ of their general deformations V' are generically reduced. We give two examples.

(1) Let $V_4 \subset \mathbb{P}^4$ be a smooth quartic 3-fold. If a hyperplane section of V_4 is a cone over a plane quartic D, then $(\mathrm{Hilb}_{1,0}\,V_4)_{\mathrm{red}}$ has D as its irreducible component. Moreover, $\mathrm{Hilb}_{1,0}\,V_4$ is non-reduced along the component ([8, II §3]).

(2) In [12], Mukai and Umemura studied a compactification $U_{22} := \overline{PSL(2)/I_{60}} \subset \mathbb{P}^{12}$ of the quotient variety of $PSL(2)$ by the icosahedral group I_{60}. It is proved that the Hilbert scheme $\mathrm{Hilb}_{1,0}\,U_{22}$ of lines in U_{22} is a double $\mathbb{P}^1$. However U_{22} has the 6-dimensional deformation space, and $\mathrm{Hilb}_{1,0}\,U'_{22}$ is generically reduced for every deformation $U'_{22} \not\cong U_{22}$ of U_{22}. (cf. Prokhorov[15]).

§3.4 Curves on a quintic 3-fold

A generic projection $C = [C_8 \subset \mathbb{P}^3]$ of canonical curves of genus 5 appears also in Voisin's example (Clemens-Kley[1]). It is proved that if a smooth quintic 3-fold $V_5 \subset \mathbb{P}^4$ contains C, then the Hilbert scheme $\mathrm{Hilb}^S_{8,5}\,V_5$ has an embedded component at $[C]$.

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