OBSTRUCTIONS TO DEFORMING SPACE CURVES LYING ON A SMOOTH QUARTIC SURFACE

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Abstract. We study the deformations of space curves $C \subset \mathbb{P}^3$ lying on a smooth quartic surface S, under the presence of a smooth rational or elliptic curve $E \subset S$. Let $\operatorname{Hilb}^{sc}\mathbb{P}^3$ denote the Hilbert scheme of smooth connected curves in $\mathbb{P}^3$. We give a sufficient condition for $\operatorname{Hilb}^{sc}\mathbb{P}^3$ to be (non)singular at $[C]$ in terms of E and an effective divisor $D := C - 4\mathbf{h}$ on S, where $\mathbf{h}$ is the class of hyperplane sections of S. As an application, we find a new class of generically non-reduced irreducible components of $\operatorname{Hilb}^{sc}\mathbb{P}^3$.

1. Introduction

Curves embedded into the projective 3-space $\mathbb{P}^3$ are called *space curves*. The problem of classifying space curves is classical and nowadays the problem is well understood (at least, theoretically) in terms of the Hilbert scheme. Let $\operatorname{Hilb}^{sc}\mathbb{P}^3$ denote the Hilbert scheme of smooth connected curves $C \subset \mathbb{P}^3$ and let $H^S_{d,g} \subset \operatorname{Hilb}^{sc}\mathbb{P}^3$ denote the subscheme parametrizing curves of degree d and genus g. For the classification, it suffices to determine the all irreducible components of $H^S_{d,g}$ for a given degree d and a given genus g. However this task is not easy in general.

On the other hand, among all space curves, the curves which are contained in a (smooth) surface $S \subset \mathbb{P}^3$ of low-degree $s \leq 4$ were intensively studied by many authors, e.g. Kleppe[7], Ellia[1], Gruson-Peskine[5], Nasu[14], etc. More recently, in the paper [8] with an appendix by Ottem, Kleppe has studied maximal families $W \subset H^S_{d,g}$ of the space curves whose general member is contained in a smooth quartic surface S. If W is an irreducible component of $H^S_{d,g}$, then the Picard number of S is at most 2. Kleppe has explicitly described non-reduced and generically smooth components in the case where $\operatorname{Pic} S$ is generated by the classes of a line and a smooth place cubic curve (cf. Remark 5.4). In this article, we study the deformations of space curves $C \subset \mathbb{P}^3$ lying on a smooth quartic surface $S \subset \mathbb{P}^3$ whose Picard group is generated by the class of hyperplane sections $\mathbf{h}$ of S and the class of a rational (or elliptic) curve $E \subset \mathbb{P}^3$ (i.e. $\operatorname{Pic} S = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E$). Since S is a $K3$ surface, E is rational ($E \simeq \mathbb{P}^1$) if and only if $E^2 = -2$, i.e., E is a (-2)-curve. Similarly E is elliptic if and only if $E^2 = 0$ (Then S is an elliptic surface and E is a fiber of an elliptic fibration $S \to \mathbb{P}^1$). We discuss the obstructions to deforming C in $\mathbb{P}^3$ and we obtain the next theorem.

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Theorem 1.1. *Let $C \subset \mathbb{P}^3$ be a smooth connected curve lying on a smooth quartic surface S. We assume that the divisor $D := C - 4\mathbf{h}$ on S is effective. Then*

(1) *If $H^1(S, D) = 0$, then $\operatorname{Hilb}^{sc} \mathbb{P}^3$ is nonsingular at $[C]$.*
(2) *If $\operatorname{Pic} S = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E$ for a (-2)-curve E and we have $E.D = -2$ and $D \neq E$, then $\operatorname{Hilb}^{sc} \mathbb{P}^3$ is singular (in fact, non-reduced) at $[C]$.*
(3) *If $D \sim mE$ for an elliptic curve E $(m \geq 2)$, then $\operatorname{Hilb}^{sc} \mathbb{P}^3$ is singular (in fact, non-reduced) at $[C]$.*

We have $h^1(S, D) = 1, m-1$ $(= h^1(\mathbb{P}^3, \mathcal{I}_C(4)))$ in the cases (2),(3) of Theorem 1.1, respectively. Here the numbers $h^1(S, D)$ are equal to the codimensions of the Hilbert scheme $\operatorname{Hilb}^{sc} \mathbb{P}^3$ in its tangent space $H^0(C, N_{C/\mathbb{P}^3})$ at $[C]$. It is known that if $D^2 \geq 0$ then we have $H^1(S, D) \neq 0$ if and only if there exists

- an effective divisor Δ such that $\Delta^2 = -2$ and $\Delta.D \leq -2$, or
- a nef and primitive divisor $F \geq 0$ such that $F^2 = 0$ and $D \sim mF$ for some $m \geq 2$

(cf. [9]).

Mumford [12] first found an example of a generically non-reduced irreducible component of the $\operatorname{Hilb}^{sc} \mathbb{P}^3$, which was constructed from a family of smooth connected curves in $\mathbb{P}^3$ lying on a smooth cubic surface. On a smooth quartic surface, a (-2)-curve and an elliptic curve play a role very similar to that for a (-1)-curve (i.e. a line) on the smooth cubic surface in Mumford's example. As an application, we construct infinitely many generically non-reduced irreducible components of $\operatorname{Hilb}^{sc} \mathbb{P}^3$.

Theorem 1.2. *Let W be a 4-maximal irreducible subset of $\operatorname{Hilb}^{sc} \mathbb{P}^3$ whose general member C is contained in a smooth quartic surface S with $\operatorname{Pic} S = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E$, where E is a smooth elliptic curve on S. If C is linearly equivalent to $4\mathbf{h}+mE$ for $m \geq 2$, then the closure $\overline{W}$ of W in $\operatorname{Hilb}^{sc} \mathbb{P}^3$ is an irreducible component of $(\operatorname{Hilb}^{sc} \mathbb{P}^3)_{\mathrm{red}}$ and $\operatorname{Hilb}^{sc} \mathbb{P}^3$ is a generically non-reduced along W.*

Let e be the degree of the elliptic curve E in $\mathbb{P}^3$. Then the degree d and genus g of C are computed as $d = me + 16$ and $g = 4me + 33$. The dimension of the irreducible component $\overline{W}$ is computed as $\dim W = g + 33 = 4me + 66 = 4d + 2$.

Let $N_{C/\mathbb{P}^3}$ denote the normal bundle of C in $\mathbb{P}^3$. It is known that there exists a one-to-one correspondence between the first order deformations of C in $\mathbb{P}^3$ and the global sections of $N_{C/\mathbb{P}^3}$. Then the obstruction $\operatorname{ob}(\varphi)$ to lifting a first order deformation $\tilde{C}$ $(\leftrightarrow \varphi \in H^0(C, N_{C/\mathbb{P}^3}))$ of a curve C in $\mathbb{P}^3$ to a second order deformation is given by the cup product

$$H^0(C, N_{C/\mathbb{P}^3}) \times H^0(C, N_{C/\mathbb{P}^3}) \xrightarrow{\cup} H^1(C, N_{C/\mathbb{P}^3}), \qquad \varphi \longmapsto \varphi \cup \varphi = \operatorname{ob}(\varphi).$$

In [11] Mukai and Nasu have developed a technique to computing this cup product and showing that it is nonzero (the theorem to show the nonzero is called the *obstructedness* *criterion*). For the proof of Theorems 1.1 and 1.2, we further develop the technique and obtain a generalization of the criterion (cf. Theorem 3.1).

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2. Hilbert scheme and Infinitesimal deformations

First we recall some basic facts on the deformation theory of embedded schemes. We work over an algebraically closed field k of characteristic $p \geq 0$. Let $V \subset \mathbb{P}^n$ be a closed subscheme of $\mathbb{P}^n$ with embedding line bundle $\mathcal{O}_V(1)$ on V, and $X \subset V$ a closed subscheme of V with the Hilbert polynomial $P(X) = \chi(X, \mathcal{O}_X(n))$. Then there exists a projective scheme H, called the *Hilbert scheme* of V, parametrizing all closed subscheme X' of V with the same Hilbert polynomial $P(X') = P(X)$ (cf. [4]). We denote by $\operatorname{Hilb} V$ the Hilbert scheme of V.

Let $\mathcal{I}_X$ and $N_{X/V} = (\mathcal{I}_X/\mathcal{I}_X^2)^\vee$ denote the ideal sheaf and the normal sheaf of X in V, respectively. The symbol $[X]$ represents the point of $\operatorname{Hilb} V$ corresponding to X. Then the tangent space of $\operatorname{Hilb} V$ at $[X]$ is well known to be isomorphic to the group $\operatorname{Hom}(\mathcal{I}_X, \mathcal{O}_X)$ and to the 0-th cohomology group $H^0(X, N_{X/V})$ of $N_{X/V}$. Every obstruction ob to deforming X in V is contained in the group $\operatorname{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$. If X is a locally complete intersection in V, then ob is contained in the subgroup $H^1(X, N_{X/V}) \subset \operatorname{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$ and we have

$$h^0(X, N_{X/V}) - h^1(X, N_{X/V}) \leq \dim_{[X]} \operatorname{Hilb} V \leq h^0(X, N_{X/V}).$$

The left hand side ($= \chi(X, N_{X/V})$) is called the *expected dimension* of $\operatorname{Hilb} V$ at $[X]$. If $H^1(X, N_{X/V}) = 0$, then the Hilbert scheme $\operatorname{Hilb} V$ is nonsingular at $[X]$ of dimension $h^0(X, N_{X/V})$. Let $D = k[t]/(t^2)$ be the ring of dual number. Then a *first order (infinitesimal) deformation* of X in V is a closed subscheme $X' \subset X \times \operatorname{Spec} D$, flat over D, and with central fiber $X'_0 = X$. By the universal property of the Hilbert scheme, there exists a one-to-one correspondence between the set of D-valued points $\psi : \operatorname{Spec} D \to \operatorname{Hilb} V$ sending 0 to $[X]$, and the set of the first order deformations of X in V.

Applying the infinitesimal lifting property of smoothness (cf. [6, Proposition 4.4,Chap. 1]) to the surjection $k[t]/(t^{n+2}) \to k[t]/(t^{n+1}) \to 0$ of artinian rings, we have

Proposition 2.1. *If* $\operatorname{Hilb} V$ *is nonsingular at* $[X]$*, then for every integer* $n \geq 1$*, every infinitesimal deformation of* X *in* V *of order* n *lifts to some infinitesimal deformation of* X *in* V *of order* $n+1$.

This fact implies that if some first order deformation of X in V does not lift to any second order deformation, then $\operatorname{Hilb} V$ is singular at $[X]$.

From now on we suppose that X is a complete intersection subscheme of V. Let $\tilde{X}$ be a first order deformation of X in V. Then there exists a global section φ of $N_{X/V}$ corresponding to $\tilde{X}$. We define a cup product $\operatorname{ob}(\varphi) \in \operatorname{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$ by

$$\operatorname{ob}(\varphi) := \varphi \cup \mathbf{e} \cup \varphi,$$

where $\mathbf{e}$ is the extension class of the standard exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_V \to \mathcal{O}_X \to 0$ on V. Then $\tilde{X}$ lifts to a second order deformation if and only if $\operatorname{ob}(\varphi)$ is zero. Since X is a locally complete intersection, $\operatorname{ob}(\varphi)$ is contained in $H^1(X, N_{X/V})$, and called the *obstruction* for φ. In what follows, we say X is *unobstructed* (resp. *obstructed*) if $\operatorname{Hilb} V$ is nonsingular (resp. singular) at $[X]$, and for an irreducible closed subset W of $\operatorname{Hilb} V$, we say $\operatorname{Hilb} V$ is *generically smooth* (resp. *generically non-reduced*) along W if $\operatorname{Hilb} V$ is nonsingular (resp. singular) at the generic point X_η of W.

3. Obstructedness criterion

In [11] Mukai and Nasu have studied the embedded deformations of a smooth curve C on a smooth projective 3-fold V under the presence of a smooth surface S such that $C \subset S \subset V$. They have given a sufficient condition for a first order infinitesimal deformation of C in V to be obstructed: if there exists a curve E on S such that $(E^2)_S < 0$ (e.g. a (-1)-curve on S) and the exterior component of φ lifts to a global section $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$, then the exterior component of $\operatorname{ob}(\varphi)$ is nonzero under a certain additional condition on E, C and v. We generalize their result in the viewpoint of the order of the pole of v along E and the self intersection number E^2.

We recall the definition of 'exterior components' and 'infinitesimal deformations with a pole', introduced in [11]. Let C be a smooth curve on a smooth projective 3-fold V. Every first order infinitesimal deformation $\tilde{C}$ of C in V determines a global section $\varphi \in H^0(C, N_{C/V})$ and the cohomology class $\operatorname{ob}(\varphi) \in H^1(C, N_{C/V})$ defined in §2, and $\tilde{C}$ lifts to a deformation over $\operatorname{Spec} k[t]/(t^3)$ if and only if $\operatorname{ob}(\varphi) = 0$.

Exterior components. Let $\pi_{C/S} : N_{C/V} \to N_{S/V}\big|_C$ be a natural projection of normal bundles. Then the projection $\pi_{C/S}$ induces the maps

$$H^i(\pi_{C/S}) : H^i(C, N_{C/V}) \to H^i(C, N_{S/V}\big|_C) \qquad (i = 0, 1)$$

on the cohomology groups. The images

$$\begin{aligned} \pi_{C/S}(\varphi) &:= H^0(\pi_{C/S})(\varphi) \\ \operatorname{ob}_S(\varphi) &:= H^0(\pi_{C/S})(\operatorname{ob}(\varphi)) \end{aligned}$$

of φ and $\mathrm{ob}(\varphi)$ by the induced maps are called the *exterior component* of φ and $\mathrm{ob}(\varphi)$, respectively. Roughly speaking, the exterior components represent the first order deformation of C in V into the direction normal to S and its obstruction, respectively.

Infinitesimal deformations with a pole. Let E be a smooth curve on S and let $m \geq 1$ be an integer. Since E is an effective divisor on S, there exists a natural short exact sequence $[0 \to \mathcal{O}_S \to \mathcal{O}_S(E) \to \mathcal{O}_E(E) \to 0] \otimes \mathcal{O}_S(mE)$ on S. We assume that the induced map

$$H^1(S, \mathcal{O}_S(mE)) \longrightarrow H^1(S, \mathcal{O}_S((m+1)E))$$

on the cohomology groups is injective for all $m \geq 1$. Let S° denote the open surface $S \setminus E \subset S$, i.e., the complement of E in S. Then there exists a natural filtration

$$H^1(S, E) \subset H^1(S, 2E) \subset \cdots \subset H^1(S, mE) \subset \cdots \subset H^1(S^\circ, \mathcal{O}_{S^\circ})$$

on $H^1(S^\circ, \mathcal{O}_{S^\circ})$. A rational section v of $N_{S/V}$ admitting a pole along E of order m, i.e., $v \in H^0(N_{S/V}(mE)) \setminus H^0(N_{S/V}(m-1)E)$ is called an *infinitesimal deformation with a pole* (along E of order m). Every infinitesimal deformation with a pole induces a first order infinitesimal deformation of the open surface $S^\circ = S \setminus E$ in $V^\circ = V \setminus E$, since there exists a natural injective map

$$H^0(S, N_{S/V}(mE)) \hookrightarrow H^0(S^\circ, N_{S^\circ/V^\circ}).$$

Let $v \in H^0(N_{S/V}(mE))$ be an infinitesimal deformation with a pole along E of order m. We consider the restriction $v\big|_E \in H^0(N_{S/V}(mE)\big|_E)$ of v to E, which is nothing but the principal part of v at E. Let ∂_E be the coboundary map of the short exact sequence

$$[0 \to \underbrace{N_{E/S}}_{\simeq \mathcal{O}_E(E)} \to N_{E/V} \to N_{S/V}\big|_E \to 0] \otimes \mathcal{O}_S(mE),$$

and let

$$\partial_E(v\big|_E) \in H^1(E, N_{E/S}(mE)) \simeq H^1(E, \mathcal{O}_E((m+1)E))$$

be the image of $v\big|_E$ by ∂_E. Then we consider the cup product

$$\partial_E(v\big|_E) \cup v\big|_E \in H^1(E, N_{S/V}((2m+1)E - C)\big|_E) \tag{3.1}$$

of $\partial_E(v\big|_E)$ with $v\big|_E$ by the natural map

$$H^1(E, \mathcal{O}_E((m+1)E)) \times H^0(E, N_{S/V}(mE - C)\big|_E) \xrightarrow{\cup} H^1(E, N_{S/V}((2m+1)E - C)\big|_E).$$

The next theorem is a generalization of [11, Theorem 2.2], in which we give a sufficient condition for a first order infinitesimal deformation $\tilde{C}$ ($\subset V \times \operatorname{Spec} k[t]/(t^2)$) of C in V to be obstructed. i.e, $\tilde{C}$ does not lift to any second order deformation $\tilde{\tilde{C}}$ ($\subset V \times \operatorname{Spec} k[t]/(t^3)$).

Theorem 3.1. *Let $\tilde{C}$ or $\varphi \in H^0(C, N_{C/V})$ be a first order infinitesimal deformation of C, and let $\pi_{C/S}(\varphi)$ be the exterior component of φ. Suppose that the image of $\pi_{C/S}(\varphi)$ in $H^0(C, N_{S/V}(mE)\big|_C)$ lifts to an infinitesimal deformation*

$$v \in H^0(S, N_{S/V}(mE)) \setminus H^0(S, N_{S/V}(m-1)E)$$

with a pole along E of order $m \geq 1$. In other words, we have

$$\pi_{C/S}(\varphi) = v\big|_C \quad \textit{in} \quad H^0(C, N_{S/V}(mE)\big|_C).$$

If moreover, the following conditions are satisfied, then the exterior component $\mathrm{ob}_S(\varphi)$ of $\mathrm{ob}(\varphi)$ is nonzero:

(a) *Let Δ be a divisor on S defined by $\Delta := C + K_V\big|_S - (m+1)E$. Then the restriction map*

$$H^0(S, \Delta) \xrightarrow{|_E} H^0(E, \Delta\big|_E)$$

is surjective, and

(b) *m is not a multiple of the character p and the cup product $\partial_E(v\big|_E) \cup v\big|_E$ defined in (3.1) is nonzero.*

Figure 1 may be useful for understanding the relation between φ and $v\big|_E$ (and $\partial(v\big|_E)$) in (b).

$$\begin{array}{ccccccccc}
H^0(N_{C/V}) & \ni & \varphi & & & & & & H^0(N_{E/V}(mE)) \\
\downarrow{\scriptstyle \pi_{C/S}} & & \downarrow & & & & & & \downarrow{\scriptstyle \pi_{E/S}(mE)} \\
H^0(N_{S/V}\big|_C) & \ni & v\big|_C & \xleftarrow{res} & v & \xrightarrow{res} & v\big|_E & \in & H^0(N_{S/V}(mE)\big|_E) \\
\bigcap & & & & \Cap & & \downarrow & & \downarrow{\scriptstyle \partial_E} \\
H^0(N_{S/V}(mE)\big|_C) & & \xleftarrow{res} & & H^0(N_{S/V}(mE)) & & \partial_E(v\big|_E) & \in & H^1(\mathcal{O}_E((m+1)E))
\end{array}$$

FIGURE 1. Relation between φ and $\partial_E(v\big|_E)$

Remark 3.2. (1) In [11, Theorem 2.2], we need some additional assumptions: E is a negative curve on S, i.e., $E^2 < 0$ on S and the order m of v along E is one.

(2) When we apply the above theorem, E is not necessarily a (-1)-curve. For example, we have an application to a $K3$ surface S, in which E is a (-2)-curve or an elliptic curve ($E^2 = 0$) (See §5).

4. Space curves lying on a low-degree surface

In this section, we recall some basic results on the deformations of space curves $C \subset \mathbb{P}^3$ and a surface $S \subset \mathbb{P}^3$ containing C. We refer to [7] and [8] for the proof of the results in this section.

First we recall the Hilbert-flag scheme $D(d,g,s)$, which parametrizes all flags $C \subset S \subset \mathbb{P}^3$ of a curves C of degree d and genus g and a surface S of degree s. We denote by $D(d,g,s)^S$ the open subscheme of $D(d,g,s)$ parametrizing flags of $C \subset S \subset \mathbb{P}^3$ with smooth C. Let $H(s)$ denote the Hilbert scheme of surfaces $S \subset \mathbb{P}^3$. Then there exists a natural diagram of the Hilbert(-flag) schemes

$$\begin{array}{ccc} D(d,g,s)^S & \xrightarrow{pr_1} & H(s) \\ {\scriptstyle pr_2}\big\downarrow & & \\ H^S_{d,g} & & \end{array} \qquad \left(\begin{array}{ccc} (C,S) & \longmapsto & S \\ \big\downarrow & & \\ C & & \end{array} \right),$$

where pr_i is the natural projection morphisms to the i-th factor ($i = 1, 2$). The tangent space A^1 of $D(d,g,s)$ at (C,S) is given by the Cartesian diagram

$$\begin{array}{ccc} A^1 & \xrightarrow{p_1} & H^0(S, N_{S/\mathbb{P}^3}) \\ {\scriptstyle p_2}\big\downarrow & \square & {\scriptstyle m}\big\downarrow \\ H^0(C, N_{C/\mathbb{P}^3}) & \xrightarrow{\pi_{C/S}} & H^0(C, N_{S/\mathbb{P}^3}\big|_C). \end{array}$$

Suppose that C and S are smooth and let $\delta : H^0(C, N_{C/\mathbb{P}^3}\big|_C) \to H^1(C, N_{C/S})$ be the coboundary map of the standard short exact sequence $0 \to N_{C/S} \to N_{C/\mathbb{P}^3} \to N_{S/\mathbb{P}^3}\big|_C \to 0$ on $\mathbb{P}^3$ and let $\alpha_{C/S}$ be the composition $\delta \circ m$ of m with δ. Then every obstruction to deforming a pair (C,S) of a curve C and a surface S with $C \subset S \subset \mathbb{P}^3$ is contained in the group

$$A^2 := \operatorname{coker} \alpha_{C/S},$$

which we call the *obstruction group* of $D(d,g,s)$ at (C,S). Moreover, there exists an exact sequence

$$\begin{aligned} (4.1) \qquad 0 &\longrightarrow H^0(S, N_{S/\mathbb{P}^3}(-C)) \longrightarrow A^1 \longrightarrow H^0(C, N_{C/\mathbb{P}^3}) \\ &\longrightarrow H^1(S, N_{S/\mathbb{P}^3}(-C)) \longrightarrow A^2 \longrightarrow H^1(C, N_{C/\mathbb{P}^3}) \\ &\longrightarrow H^2(S, N_{S/\mathbb{P}^3}(-C)) \end{aligned}$$

of cohomology groups, which connect the tangent space and the obstruction group of $H^S_{d,g}$ with those of $D(d,g,s)^S$. We deduce the following facts from the exact sequence (4.1).

Lemma 4.1 (cf. [7, 8]). (1) *If $H^1(S, N_{S/\mathbb{P}^3}(-C)) = 0$, then the first projection $pr_1 : D(d,g,s) \to H^S_{d,g}$ is smooth at (C,S).*

(2) *If $d > s^2$ then we have $H^0(S, N_{S/\mathbb{P}^3}(-C)) = 0$ and pr_1 is a locally embedding at (C,S).*

(3) *We have*

$$\begin{aligned} (4.2) \qquad \dim A^1 - \dim A^2 &= \chi(S, N_{S/\mathbb{P}^3}) + \chi(C, N_{C/S}) \\ &= (4-s)d + g + \binom{s+3}{3} - 2. \end{aligned}$$

The equation (4.2) represents the expected dimension of the Hilbert-flag scheme $D(d,g,s)$ at (C,S). Since we have $N_{C/S} \sim -K_S\big|_C + K_C$ and $K_S \sim \mathcal{O}_S(s-4)$, by the Serre duality, we have

$$H^1(C, N_{C/S}) \simeq H^1(C, -K_S\big|_C + K_C) \simeq H^0(C, \mathcal{O}_C(s-4))^\vee.$$

Then $H^1(C, N_{C/S})$ vanishes if $s \le 3$ and is of dimension one if $s = 4$. It is known that if $s \ge 4$ and C is not a complete intersection on S, then $\alpha_{C/S}$ is not a zero map as a consequence of the infinitesimal Noether Lefschets theorem (cf. [3, Theorem 4.f.3]). Hence $\alpha_{C/S}$ is surjective in the case $s = 4$. Therefore we obtain the first statement of the next proposition.

Proposition 4.2 (cf. [8, Proposition 1.2],[7])**.** *If $s \le 4$, $d > s^2$ and C is not a complete intersection for $s = 4$, then we have*

1. *$D(d,g,s)$ is nonsingular of the expected dimension* (4.2) *at* (C,S).
2. *If $H^1(S, N_{S/\mathbb{P}^3}(-C)) = 0$ or $H^2(S, N_{S/\mathbb{P}^3}(-C)) = 0$, then $H^S_{d,g}$ is nonsingular at $[C]$.*

Proof. If $H^1(S, N_{S/\mathbb{P}^3}(-C)) = 0$, then pr_1 is smooth at (C,S) by Lemma 4.1(1). Thus we get the smoothness of $H^S_{d,g}$ at $[C]$ from the first statement. If $H^2(S, N_{S/\mathbb{P}^3}(-C)) = 0$, then we get $H^1(C, N_{C/\mathbb{P}^3}) = 0$ by the exact sequence (4.1). □

Let S be a smooth surface of degree $s \le 4$, $C \subset S$ a smooth connected curve of degree $d > s^2$. Suppose that C is not complete intersection. Then the Hilbert-flag scheme is smooth at (C,S) by Proposition 4.2(1) and there exists a unique irreducible component $\mathcal{W}$ of $D(d,g,s)$ passing through (C,S). Let $W \subset H^S_{d,g}$ be the image of $\mathcal{W}$ by pr_1. Then W is an irreducible closed subset of $H^S_{d,g}$, whose general member C is contained in a smooth surface S of degree s.

Given an irreducible subset U of $H^S_{d,g}$, $s(U)$ denote the minimal degree of surfaces containing a general member of U. An irreducible closed subset U of $H^S_{d,g}$ is called s-maximal if $s(U) = s$ and if it satisfies $s(V) > s$ for any irreducible closed subset V strictly containing U. In this case, we say U is *s-maximal family or subset* of $H^S_{d,g}$. Here $W = pr_1(\mathcal{W})$ is clearly a s-maximal family of $H^S_{d,g}$.

5. Obstructions to deforming curves on a smooth quartic surface

Now we assume $\operatorname{char} p = 0$. Let C be a smooth connected curve on a smooth quartic surface S. As is well known, S is a $K3$ surface, i.e., K_S is trivial and $H^1(S, \mathcal{O}_S) = 0$. If S is general, then $\operatorname{Pic} S$ is generated by the class $\mathbf{h}$ of hyperplane sections of S. Thus C is linearly equivalent to a multiple $n\mathbf{h}$ for some $n \ge 1$. In the case, C is a complete intersection of S and a surface of degree n in $\mathbb{P}^3$. Then in particular, C is arithmetically Cohen-Macaulay, and we see that C is unobstructed, thanks to a result of Ellingsrud [2].

In what follows, we assume that S is not general and C is not a complete intersection with another surface. Let d and g be the degree and the genus of C, respectively. Suppose that $D := C - 4\mathbf{h}$ is effective. Then we have $d > 16 = 4^2$ and C belongs to a unique 4-maximal family $W \subset H^S_{d,g}$ and the dimension of W is equal to $g + 33$ by (4.2). Since $H^0(C, N_{C/\mathbb{P}^3})$ represents the tangent space of the Hilbert scheme $H^S_{d,g}$ at $[C]$, there exists a natural inequality

$$\dim W \leq \dim_{[C]} H^S_{d,g} \leq h^0(C, N_{C/\mathbb{P}^3}). \tag{5.1}$$

of dimensions. Furthermore, we have

Lemma 5.1. (1) *$H^S_{d,g}$ is nonsingular at $[C]$ if and only if $\dim_{[C]} H^S_{d,g} = h^0(C, N_{C/\mathbb{P}^3})$.*
(2) *W is an irreducible component of $(H^S_{d,g})_{\mathrm{red}}$ if and only if $\dim W = \dim_{[C]} H^S_{d,g}$.*

There exists a short exact sequence $[0 \to \mathcal{I}_S \to \mathcal{I}_C \to \mathcal{O}_S(-C) \to 0] \otimes \mathcal{O}_{\mathbb{P}^3}(4)$ on $\mathbb{P}^3$. Then by $\mathcal{I}_S \simeq \mathcal{O}_{\mathbb{P}^3}(-4)$, we have

$$h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = h^1(S, 4\mathbf{h} - C) = h^1(S, D).$$

By using the Riemann-Roch theorem on C and S, we can compute that

$$h^1(S, D) = h^0(C, N_{C/\mathbb{P}^3}) - \dim W. \tag{5.2}$$

Thus if $H^1(S, D) = 0$, then we have

$$\dim W = \dim_{[C]} H^S_{d,g} = h^0(C, N_{C/\mathbb{P}^3}),$$

and thereby we have proved Theorem 1.1 (1). On the other hand, if we have $H^1(S, D) \neq 0$ and $D^2 \geq 0$, then there exists

- an effective divisor Δ such that $\Delta^2 = -2$ and $\Delta.D \leq -2$, or
- a nef and primitive divisor $F \geq 0$ such that $F^2 = 0$ and $D \sim mF$ for some $m \geq 2$

by the following theorem.

Theorem 5.2 (a special case of [9])**.** *Let X be a $K3$ surface, L a line bundle on X. Suppose that $L > 0$ and $L^2 \geq 0$. Then $H^1(X, L) \neq 0$ if and only if*

(1) *there exists an effective divisor Δ such that $\Delta^2 = -2$ and $\Delta.L \leq -2$, or*
(2) *$L \sim nF$ $(n \geq 2)$ for a nef and primitive divisor $F \geq 0$ with $F^2 = 0$.*

Once we find a smooth quartic surface containing smooth curves, e.g. a line, a smooth conic, or a smooth plane cubic etc., then we can find also a smooth quartic surface of Picard number 2 containing them by the following result due to Mori[10].

Proposition 5.3. *If there exists a smooth quartic surface S_0 containing a smooth curve E_0 of degree d and genus g, then there also exists a smooth quartic surface S containing a smooth curve E of the same degree and genus as E_0 such that $\operatorname{Pic} S = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E$.*

5.1. **Outline of the proof of Theorem 1.1.** We give an outline of the proof of Theorem 1.1. We have already given the proof of (1). Now we give an outline for (2). Suppose that $\operatorname{Pic} S = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E$, where E is a (non-complete intersection) smooth rational curve on S. Then C is linearly equivalent to $a\mathbf{h}+bE$ for some $a, b \in \mathbb{Z}$. The intersection matrix of S is given by

$$\begin{pmatrix} \mathbf{h}^2 & \mathbf{h}.E \\ \mathbf{h}.E & E^2 \end{pmatrix} = \begin{pmatrix} 4 & e \\ e & -2 \end{pmatrix},$$

where e is the degree of E in $\mathbb{P}^3$. By the assumption that $E.D = -2$ and $D \neq E$, $D = C - 4\mathbf{h}$ is linearly equivalent to a divisor defined by

$$\begin{cases} k(2\mathbf{h} + eE) + E & \text{(when } e \text{ is odd)} \\ k(\mathbf{h} + e'E) + E & \text{(when } e = 2e' \text{ is even)} \end{cases}$$

for some integer $k \geq 1$. We compute that $H^i(S, D - E) = 0$ for $i = 1, 2$ by using e.g. the Kawamata-Viehweg vanishing theorem. Then it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D-E) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_E(D) \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow 0,$$

that we have $h^1(S, D) = 1$.

Let t_W denote the tangent space of the maximal family $W \subset H^S_{d,g}$ containing C, which is a subspace of the tangent space $H^0(C, N_{C/\mathbb{P}^3})$ of the Hilbert scheme $H^S_{d,g}$. Then by (5.2), there exists a global section φ of $N_{C/\mathbb{P}^3}$ which is not contained in t_W. Then we can apply the obstructedness criterion (Theorem 3.1) to φ and we obtain $\operatorname{ob}_S(\varphi) \neq 0$. This implies that $H^S_{d,g}$ is singular at $[C]$ by Proposition 2.1. Then the inequality (5.1) shows that $\dim H^S_{d,g} = \dim W$ and hence W is an irreducible component of $H^S_{d,g}$. Since C is general member of W, $H^S_{d,g}$ is nowhere reduced along W. Thus we have completed the proof of Theorem 1.1(2).

Finally we give an outline of the proof of (3). Let E be a smooth elliptic curve on S and suppose that $D = C - 4\mathbf{h} \sim (m+1)E$ for $m \geq 1$. Then we have $h^1(S, D) = h^1(S, mE) = m$ and hence by (5.2), the tangent space $H^0(C, N_{C/\mathbb{P}^3})$ of $H^S_{d,g}$ at $[C]$ is greater than $\dim W$ by m. Then by Lemma 5.1, it suffices to show that $\dim_{[C]} H^S_{d,g} = \dim W$, because this implies also that $H^S_{d,g}$ is singular at $[C]$ by the same lemma. Let φ be an arbitrary global section of $N_{C/\mathbb{P}^3}$, which is not contained in the tangent space t_W of W. Since $N_{S/\mathbb{P}^3}(mE - C) \sim 4\mathbf{h} + mE - C \sim -D + mE = -E$, we have $H^1(S, N_{S/\mathbb{P}^3}(mE - C)) = 0$. Then the exterior component $\pi_S(\varphi)$ (cf. §3) lifts to an infinitesimal deformation v with a pole (cf. §3) along E of order k for some $1 \le k \le m$ by the following diagram.

$$\begin{array}{ccccc}
\varphi & \longmapsto & \pi_S(\varphi) & & H^0(S, N_{S/\mathbb{P}^3}(mE)) \ni {}^\exists v \\
\Cap & & \Cap & & \downarrow \\
H^0(C, N_{C/\mathbb{P}^3}) & \xrightarrow{\pi_{C/S}} & H^0(C, N_{S/\mathbb{P}^3}\big|_C) & \subset & H^0(C, N_{S/\mathbb{P}^3}(mE)\big|_C) \\
& & \downarrow & & \downarrow \\
& & H^1(S, N_{S/\mathbb{P}^3}(-C)) \simeq k^m & \to & H^1(S, N_{S/\mathbb{P}^3}(mE - C)) = 0
\end{array}$$

Applying the obstructedness criterion (Theorem 3.1), we finally obtain $\mathrm{ob}_S(\varphi) \neq 0$. Thus we have proved Theorem 1.1(3) and Theorem 1.2.

Remark 5.4. In [8] Kleppe (with Ottem) has also studied the deformations of space curves lying on a smooth quartic surface $S \subset \mathbb{P}^3$. A quartic surface which they have considered is a smooth quartic surface $S \subset \mathbb{P}^3$ whose Picard group $\operatorname{Pic} S$ is generated over $\mathbb{Z}$ by the classes of a line E and a smooth plane cubic F ($\sim \mathbf{h} - E$). Then E and F satisfy $E^2 = -2$, $F^2 = 0$, $E.F = 3$. Let $C \subset \mathbb{P}^3$ be a smooth connected curve of degree d and genus g lying on S. Suppose that $D := C - 4\mathbf{h}$ is effective and C is not a complete intersection. Then C belongs to a unique 4-maximal family W of the Hilbert scheme $H^S_{d,g}$. Furthermore $\dim W = g + 33$. They have proved the following:

(1) If $D.E \ge -1$ and $D.F > 0$, then W is a generically smooth irreducible component of $H^S_{d,g}$, and
(2) If $D.E \le -2$ and g is sufficiently large*, then W is a generically non-reduced irreducible component of $H^S_{d,g}$.

Let $C \sim aE + bF$ with $a, b \ge 0$ (we denote $C \equiv (a, b)$ for short). Then the degree d and the genus g of C is computed as $d = a + 3b$ and $g = 3ab - a^2 + 1$, respectively. Their result is illustrated by Table 1, where k denote a non-negative integer, and W denote the maximal family containing C.

Cases	[I]	[II]	[III]	[IV]
$C \equiv (a, b)$	$3b - 2a \ge 3$	$(8 + 3k, 6 + 2k)$	$(10 + 3k, 7 + 2k)$	$(15 + 3k, 10 + 2k)$
$C.E = 3b - 2a$	≥ 3	2	1	0
$D.E$	≥ -1	-2	-3	-4
$h^1(S, D)$	0	1	2	4
W	smooth component	non-reduced component		

TABLE 1. Known cases

Since F is a smooth elliptic curve, by Theorem 1.2, we have Table 2, which complements the result obtained by Kleppe (with Ottem) (cf. [8, Remark 4.1]). If $C \equiv (4, 5)$, then

*More precisely, they proved the result for the degree d and the genus g satisfying $g > \min\{G(d, 5) - 1, d^2/10 + 21\}$ and $d \ge 21$. Here $G(d, 5)$ denotes the maximum genus of curves of degree d not contained in degree-4 surface.

Case	[V]
$C \equiv (a,b)$	$(4, 4+k)$ with $k \geq 2$
$C.F$	12
$h^1(S,D)$	$k-1$
W	non-reduced component

Table 2. New case

we have $H^1(S,D) = H^1(S,E) = 0$ and hence W is a generically smooth irreducible component of $H^S_{d,g}$. Therefore for a smooth quartic surface S containing a line E and a plane cubic F with $\operatorname{Pic} S = \mathbb{Z}E \oplus \mathbb{Z}F$, and a smooth connected curve $C \subset S$ of degree d and genus g, we have determined the generic smoothness of $H^S_{d,g}$ along the 4-maximal family $W \subset H^S_{d,g}$ containing C, except for a finite number of pairs (d,g) of small d and small g.

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