Introduction to Hilbert schemes of curves on a 3-fold

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§1 Introduction

Hilbert scheme

We work over a field $k = \bar{k}$ with $\mathrm{char}\, k = 0$.

$V \subset \mathbb{P}^n$: a closed subscheme.
$\mathcal{O}_V(1)$: a very ample line bundle on V.
$X \subset V$: a closed subscheme.
$P = P(X) = \chi(X, \mathcal{O}_X(n))$: the Hilbert polynomial of X.

Then there exists a proj. scheme H, called the Hilbert scheme of V, parametrizing all closed subschemes X' of V with the same Hilbert poly. P as X.

Theorem (Grothendieck'60)

There exists a proj. scheme H and a closed subscheme $W \subset V \times H$ (universal subscheme), flat over H, such that

1. the fibers $W_h \subset W$ over a closed point $h \in H$ are closed subschemes of V with the same Hilb. poly. $P(W_h) = P$,
2. For any scheme T and a closed subscheme $W' \subset V \times T$ with the above prop. ①, there exists a unique morphism $\varphi : T \to H$ such that $W' = W \times_H T$ as a subscheme of $V \times T$ (the universal property of H).

Notation

$$\begin{array}{rl} \mathrm{Hilb}\, V & = \text{the (full) Hilbert scheme of } V \\ \bigcup \text{open} & \\ \mathrm{Hilb}^{sc} V : & = \{\text{smooth connected curves } C \subset V\} \\ \text{closed} \bigcup \text{open} & \\ \mathrm{Hilb}^{sc}_{d,g} V : & = \{\text{curves of degree degree } d \text{ and genus } g\} \end{array}$$

$(d := \deg \mathcal{O}_C(1))$

Hilbert scheme of space curves

$V = \mathbb{P}^3$: the projective 3-space over k
$C \subset \mathbb{P}^3$: a closed subscheme of $\mathbf{dim} = 1$

$d(C)$: degree of C $(= \sharp(C \cap \mathbb{P}^2))$
$g(C)$: arithmetic genus of C

We study the Hilbert scheme of space curves:

$$\begin{aligned} H_{d,g} &:= \mathbf{Hilb}^{sc}_{d,g}\, \mathbb{P}^3 \\ &= \left\{ C \subset \mathbb{P}^3 \;\middle|\; \begin{array}{l} \text{smooth and connected} \\ d(C) = d \text{ and } g(C) = g \end{array} \right\} \end{aligned}$$

Why we study $H_{d,g}$?

Some reasons are:

- For every smooth curve C, there exists a curve $C' \subset \mathbb{P}^3$ s.t. $C' \simeq C$.
- $\mathbf{Hilb}^{sc}\, \mathbb{P}^3 = \bigsqcup_{d,g} H_{d,g}$
- More recently, the classification of the space curves has been applied to the study of bir. automorphism

$$\Phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

(for the construction of Sarkisov links [Blanc-Lamy,2012]).

Some basic facts

- If $g \leq d - 3$, then $H_{d,g}$ is irreducible [Ein,'86] and $H_{d,g}$ is generically smooth of expected dimension $4d$.
- In general, $H_{d,g}$ can become reducible, e.g $H_{9,10} = W_1^{(36)} \sqcup W_2^{(36)}$ [Noether].
- the Hilbert scheme of arith. Cohen-Macaulay (ACM, for short) curves are smooth [Ellingsrud, '75].

$$C \subset \mathbb{P}^3\text{: ACM} \overset{\text{def}}{\iff} H^1(\mathbb{P}^3, \mathcal{I}_C(l)) = 0 \text{ for all } l \in \mathbb{Z}$$

- $H_{d,g}$ can have many generically non-reduced irreducible components, e.g. [Mumford'62], [Kleppe'87], [Ellia'87], [Gruson-Peskine'82], etc.

Infinitesimal property of the Hilbert scheme

V: a smooth projective variety over k
$X \subset V$: a closed subscheme of V
$\mathcal{I}_X$: the ideal sheaf defining X in V
$N_{X/V}$: the normal sheaf of X in V

Fact (Tangent space and Obstruction group)

1. The tangent space of $\mathbf{Hilb}\, V$ at $[X]$ is isomorphic to $\mathbf{Hom}(\mathcal{I}_X, \mathcal{O}_X) \simeq H^0(X, N_{X/V})$
2. Every obstruction $\mathbf{ob}$ to deforming X in V is contained in the group $\mathbf{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$. If X is a locally complete intersection in V, then $\mathbf{ob}$ is contained in $H^1(X, N_{X/V})$

If X is a loc. comp. int. in V, then we have the following inequalities:

Fact

1. We have

$$h^0(X, N_{X/V}) - h^1(X, N_{X/V}) \leq \dim_{[X]} \operatorname{Hilb} V \leq h^0(X, N_{X/V}).$$

2. In particular, if $H^1(X, N_{X/V}) = 0$, then $\operatorname{Hilb} V$ is nonsingular at $[X]$ of dimension $h^0(X, N_{X/V})$.

What is Obstruction?

$(R, \mathfrak{m})$: a local ring with residue field k.
R is a regular loc. ring if $\mathrm{gr}_{\mathfrak{m}} R := \bigoplus_{l=0}^{\infty} \mathfrak{m}^l/\mathfrak{m}^{l+1}$ is isom. to a polynomial ring over k.
X: a scheme X of finite type over k.
X is nonsingular at $x \Longleftrightarrow \mathcal{O}_{x,X}$ is a regular loc. ring.

Proposition (infinitesimal lifting property of smoothness)

R is a regular local ring if and only if for any surjective homo. $\pi : A' \to A$ of Artinian rings A, A', a ring homo. $u : R \to A$ lifts to $u' : R \to A'$.

$X(A) = \{f : \mathrm{Spec}\, A \to X\}$: the set of A-valued points of X.

X is nonsingular $\iff$ the map $X(A') \to X(A)$ is surjective for any surjection $u : A' \to A$ of Artinian rings.

If X is singular, then the map $X(A') \to X(A)$ is not surjective in general.

There exists a vector space V over k (called obstruction group) with the following property:
for any surjection $\pi : A' \to A$ of Artinian rings and $u : R \to A$, there exists an element $\mathbf{ob}(u, A') \in V$ and

$$\mathbf{ob}(u, A') = 0 \iff u \text{ lifts to } u' : R \to A'$$

Here $\mathbf{ob}(u, A')$ is called the obstruction for u.

First order deformation

$X \subset V$: a closed subscheme of V.
T: a scheme over k

Definition

A deformation of X in V over T is a closed subscheme $X' \subset V \times T$, flat over T, with $X_0 = X$.

A deformation of X over the ring of dual number $D := k[t]/(t^2)$ is called a first order deformation of X in V.
By the univ. prop. of the Hilb. sch., there exists a one-to-one correspondence between

1. D-valued pts $\mathbf{Spec}\, D \to \mathbf{Hilb}\, V$ sending $\mathbf{0} \mapsto [X]$.
2. first order deformations of X in V

Applying the infinitesimal lifting prop. of smoothness to the surjection

$$k[t]/(t^3) \to k[t]/(t^2) \to 0,$$

we have

Proposition

If $\mathbf{Hilb}\, V$ is nonsingular at $[X]$, then every first order deformation of X in V lifts to a (second) order deformation of X in V over $k[t]/(t^3)$.

$W \subset \mathbf{Hilb}\, V$: an irreducible closed subset of $\mathbf{Hilb}\, V$.
$[X] \in W$: a closed point of W
$X_\eta \in W$: the generic point of W

Definition

- We say X is unobstructed (resp. obstructed) (in V) if $\mathbf{Hilb}\, V$ is nonsingular (resp. singular) at $[X]$.
- We say $\mathbf{Hilb}\, V$ is generically smooth (resp. generically non-reduced) along W if $\mathbf{Hilb}\, V$ is nonsingular (resp. singular) at X_η.

Mumford's example (a pathology)

$S \subset \mathbb{P}^3$: a smooth cubic surface ($\simeq \mathbf{Blow}_{6\text{ pts}}\, \mathbb{P}^2$)
$h = S \cap \mathbb{P}^2$: a hyperplane section
E: a line on S
There exists a smooth connected curve

$$C \in |4h + 2E| \subset S \subset \mathbb{P}^3,$$

of degree $\mathbf{14}$ and genus $\mathbf{24}$.
Then C is parametrized by a locally closed subset

$$W = W^{(56)} \subset H_{14,24} \subset \mathrm{Hilb}^{sc}\, \mathbb{P}^3$$

of the Hilbert scheme.

The locally closed subset $W^{(56)}$ fits into the diagram

$$\overline{\left\{ C \subset \mathbb{P}^3 \,\middle|\, \begin{array}{c} C \subset {}^{\exists}S \text{ (smooth cubic)} \\ \text{and } C \sim 4h + 2E \end{array} \right\}} =: W^{(56)} \subset H_{14,24}$$

$$\downarrow \mathbb{P}^{39}\text{-bundle}$$

$$\left(\begin{array}{c} \text{family of smooth} \\ \text{cubic surfaces} \end{array} \right) =: U \underset{\text{open}}{\subset} |\mathcal{O}_{\mathbb{P}^3}(3)| \simeq \mathbb{P}^{19},$$

where we have $\dim |\mathcal{O}_S(C)| = 39$ and $h^0(N_{C/\mathbb{P}^3}) = 57$.

$H^0(N_{C/\mathbb{P}^3})$ = the tangent space of $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ at $[C]$.
We have the following inequalities:

$$56 = \dim W \leq \dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^3 \leq h^0(N_{C/\mathbb{P}^3}) = 57.$$

Thus we have a dichotomy between (A) and (B):

- (A) $\overline{W}$ is an irred. comp. of $(\mathrm{Hilb}^{sc}\,\mathbb{P}^3)_{\mathrm{red}}$.
  $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ is generically non-reduced along $\overline{W}$.
- (B) There exists an irred. comp. $W' \supsetneq W$.
  $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ is generically smooth along $\overline{W}$.

Which? $\rightsquigarrow$ The answer is (A). (It suffices to prove $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ is singular at the generic point $[C]$ of W. We will see later in §2)

History

Later many non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ were found by Kleppe['85], Ellia['87], Gruson-Peskine['82], Fløystad['93] and Nasu['05].
Moreover, to the question "How bad can the deformation space of an object be?", Vakil['06] has answered that

Law (Murphy's law in algebraic geometry)

Unless there is some a priori reason otherwise, the deformation space may be as bad as possible.

A naive question

Nowadays non-reduced components of Hilbert schemes are not seldom. However,

Question

What is/are the most important reason(s) (if any) for their existence?

Our answer is the following: (at least in Mumford's example,) a (-1)-curve E (i.e. $E \simeq \mathbb{P}^1$, $E^2 = -1$) on the (cubic) surface is the most important.

A generalization of Mumford's ex.

Theorem (Mukai-Nasu'09)

V: a smooth projective 3-fold. Suppose that

1. there exists a curve $E \simeq \mathbb{P}^1 \subset V$ s.t. $N_{E/V}$ is generated by global sections,
2. there exists a smooth surface S s.t. $E \subset S \subset V$, $(E^2)_S = -1$ and $H^1(N_{S/V}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathrm{Hilb}^{sc} V$ has infinitely many generically non-reduced components.

In Mumford's ex., $V = \mathbb{P}^3$, S: a smooth cubic, E: a line.

Examples

We have many ex. of generically non-reduced components of $\mathbf{Hilb}^{sc} V$ for uniruled 3-folds V.

Ex.

1. Let V be a Fano 3-fold and let $-K_V = H + H'$, where H, H': ample. $^\exists S \in |H|$ (smooth).
   If $S \not\simeq \mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, then there exists a (-1)-$\mathbb{P}^1$ E on S.
2. Let $V \xrightarrow{\pi} F$ be a $\mathbb{P}^1$-bundle over a smooth surface F with $p_g(F) = 0$. Let S_1 be a section of π and A a sufficiently ample divisor on F. Then there exists a smooth surface $S \in |S_1 + \pi^* A|$. Take a fiber E of $S \to F$.

§2 Infinitesimal analysis of the Hilbert scheme

In the analysis of Mumford's ex., we develop some techniques to computing the obstruction to deforming a curve on a uniruled 3-fold ("obstructedness criterion").

Setting:
V: a uniruled 3-fold
S: a surface
E: (-1)-curve on S
C: a curve on S
with $C, E \subset S \subset V$

Obst. Criterion $\Longrightarrow$

Non-reduced components of $\mathrm{Hilb}^{sc} V$

Obstructions and Cup products

$\tilde{C} \subset V \times \operatorname{Spec} k[t]/(t^2)$:
a first order (infinitesimal) deformation of C in V
(i.e.,a tangent vector of $\operatorname{Hilb} V$ at $[C]$)

$$\begin{array}{ccccc} \tilde{C} & \in & \{\text{1st ord. def. of } C\} & & \\ \updownarrow & & \updownarrow {}^{\exists}\text{1-to-1} & & \\ \alpha & \in & \operatorname{Hom}(\mathcal{I}_C, \mathcal{O}_C) & & (\simeq H^0(N_{C/V})) \end{array}$$

Define the cup product $\operatorname{ob}(\alpha)$ by

$$\operatorname{ob}(\alpha) := \alpha \cup \mathbf{e} \cup \alpha \in \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_C),$$

where $\mathbf{e} \in \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{I}_C)$ is the ext. class of an exact sequence $0 \to \mathcal{I}_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0$.

Fact

A first order deformation $\tilde{C}$ lifts to a deformation over $\mathbf{Spec}\, k[t]/(t^3)$ if and only if $\mathbf{ob}(\alpha) = 0$.

Remark

- If $\mathbf{ob}(\alpha) \neq 0$, then $\mathbf{Hilb}\, V$ is singular at $[C]$.
- If C is a loc. complete intersection in V, then $\mathbf{ob}(\alpha)$ is contained in the small group $H^1(C, N_{C/V})$ $(\subset \mathbf{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$.

Exterior components

Let $C \subset S \subset V$ be a flag of a curve, a surface and a $\mathbf{3}$-fold (all smooth), and let $\pi_{C/S} : N_{C/V} \to N_{S/V}\big|_C$ be the natural projection.

Definition

Define the *exterior component* of α and $\mathbf{ob}(\alpha)$ by

$$\begin{aligned} \pi_S(\alpha) &:= H^0(\pi_{C/S})(\alpha) \\ \mathbf{ob}_S(\alpha) &:= H^1(\pi_{C/S})(\mathbf{ob}(\alpha)), \end{aligned}$$

respectively.

Infinitesimal deformation with pole

Let $E \subset S \subset V$ be a flag of a curve, a surface and a 3-fold.

Definition

A rational section v of $N_{S/V}$ admitting a pole along E, i.e.

$$v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V}),$$

is called an infinitesimal deformation with a pole.

Remark (an interpretation)

Every inf. def. with a pole induces a 1st ord. def. of the open surface $S^\circ = S \setminus E$ in $V^\circ = V \setminus E$ by the map

$$H^0(N_{S/V}(E)) \hookrightarrow H^0(N_{S^\circ/V^\circ})$$

Obstructedness Criterion

Now we are ready to give a sufficient condition for a first order infinitesimal deformation of $\tilde{C}$ ($\subset V \times \operatorname{Spec} k[t]/(t^2)$) of C in V to be obstructed. i.e, $\tilde{C}$ does not lift to any second order deformation $\tilde{\tilde{C}}$ ($\subset V \times \operatorname{Spec} k[t]/(t^3)$).

Condition (☆)

We consider $\alpha \in H^0(N_{C/V})$ satisfying the following condition (☆): the ext. comp. $\pi_S(\alpha)$ of α lifts to an inf. def. with a pole along E, say v, and its restriction $v\big|_E$ to E does not belong to the image of the map $\pi_{E/S}(E) := \pi_{E/S} \otimes \mathcal{O}_S(E)$.

$$
\begin{array}{ccccccc}
H^0(N_{C/V}) & \ni \alpha & & & & & H^0(N_{E/V}(E)) \\
\Big\downarrow \pi_{C/S} & \Big\downarrow & & & & & \Big\downarrow \pi_{E/S}(E) \\
H^0(N_{S/V}\big|_C) & \ni \pi_S(\alpha) & \overset{\text{res.}}{\longleftarrow} & v & \overset{\text{res.}}{\longmapsto} & v\big|_E \in & H^0(N_{S/V}(E)\big|_E) \\
 & (=v\big|_C) & & & & & \\
\cap & & & \Cap & & & \\
H^0(N_{S/V}(E)\big|_C) & \overset{\text{res.}}{\longleftarrow} & & H^0(N_{S/V}(E)) & & &
\end{array}
$$

Theorem (Mukai-Nasu'09)

Let $C, E \subset S \subset V$ be as above. Suppose that $E^2 < 0$ on S, and let $\alpha \in H^0(N_{C/V})$ satisfy (☆). If moreover,

1. Let $\Delta := C + K_V\big|_S - 2E$ on S. Then

$$(\Delta \cdot E)_S = 2(-E^2 + g(E) - 1) \qquad (2.1)$$

2. the res. map $H^0(S, \Delta) \to H^0(E, \Delta\big|_E)$ is surjective,

then we have $\mathbf{ob}_S(\alpha) \neq 0$.

Remark

If E is a (-1)-$\mathbb{P}^1$ on S, then the RHS of (2.1) is equal to 0.

How to apply Obstructedness Criterion

(Mumford's ex. $V = \mathbb{P}^3$)
Every general member $C \subset \mathbb{P}^3$ of Mumford's ex. $W^{(56)} \subset \mathrm{Hilb}^{sc}\,\mathbb{P}^3$ is contained in a smooth cubic surface S and $C \sim 4h + 2E$ on S (E: a line, h: a hyp. sect.).
Let t_W denote the tangent space of W at $[C]$ ($\dim t_W = \dim W = 56$).
Then there exists a first order deformation

$$\tilde{C} \longleftrightarrow \alpha \in H^0(C, N_{C/\mathbb{P}^3}) \setminus t_W.$$

of C in $\mathbb{P}^3$.

Claim

$\mathrm{ob}(\alpha) \neq 0$.

Proof.

Since $H^1(N_{S/\mathbb{P}^3}(E-C)) = 0$, the ext. comp. $\pi_{C/S}(\alpha) \in H^0(N_{S/\mathbb{P}^3}\big|_C)$ of α has a lifts to a rational section $v \in H^0(N_{S/\mathbb{P}^3}(E))$ on S (an inf. def. with a pole). By the key lemma below, the restriction $v\big|_E$ to E is not contained $\mathrm{im}\, \pi_{E/S}(E)$. Since $C \sim 4h + 2E = -K_{\mathbb{P}^3}\big|_S + 2E$, the divisor Δ is zero. Thus the condition (1) and (2) are both satisfied. □

Lemma (Key Lemma)

Since C is general, the finite scheme $Z := C \cap E$ of length 2 is not cut out by any conic in $|h - E| \simeq \mathbb{P}^1$ on S.

§3 Obstruction to deforming curves on a quartic surface

Expectation

Let

$$C \subset S \subset V$$

be a flag of a curve, a surface, a 3-fold.

We study the deformation of C in V with a help of the intermediate surface S and rational curves $E \simeq \mathbb{P}^1$ on S.

Expectation

- Negative curves E ($E^2 < 0$) on S control the deformations of C in V.
- The obstructedness of C follows from the geometry of S and E, C.

We study the deformation of space curves

$$C \subset \mathbb{P}^3$$

under the assumption

Assumption

C is contained in a smooth quartic surface $S \subset \mathbb{P}^3$.

Here S is a K3 surface.
$\rho := \rho(S)$: the Picard number of S.
$\mathbf{h} = \mathcal{O}_S(1) \in \operatorname{Pic} S$: the cls. of hyp. section of S.

Another assumption

If S is general, then $\rho = 1$. Then $C \sim n\mathbf{h}$ for some $n \in \mathbb{N}$, i.e., C is a comp. int. on S, and hence unobstructed (ACM).

Assume that

Assumption

There exists a rational curve $E \simeq \mathbb{P}^1$ on S.

For an irred. curve $E \subset S$, we have

$$E \simeq \mathbb{P}^1 \iff E^2 = -2. \qquad ((-2)\text{-curve})$$

Mori's result

Theorem (Mori'84)

If there exists a smooth curve $E_0 \not\sim n\mathbf{h}$, on a smooth quartic surface S_0, then there exists a smooth curve E on a (general) smooth quartic surface S of the same degree and genus as E_0 satisfying

$$\mathrm{Pic}(S) = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E.$$

By Mori's result, we may assume that $\rho(S) = 2$ and

$$\mathbf{Pic}(S) = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E$$

for studying the deformation of $C \subset S$ in $\mathbb{P}^3$.

Let e $(= \mathbf{h} \cdot E)$ be the degree of E. Then the intersection matrix on S is given by

$$\begin{pmatrix} \mathbf{h}^2 & \mathbf{h} \cdot E \\ \mathbf{h} \cdot E & E^2 \end{pmatrix} = \begin{pmatrix} 4 & e \\ e & -2 \end{pmatrix}.$$

Mori cone of smooth K3 surface ($\rho = 2$)

X: a smooth K3 surface.

$\mathbf{NE}(X) := \left\{ \sum a_i [C_i] \;\middle|\; C_i\text{: irred. curve on } X,\ a_i \geq 0 \right\}$

$\overline{\mathbf{NE}(X)} = \overline{\mathbf{Eff}(X)} \subset \mathbf{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (Mori cone of X)

$\rho = 2 \Longrightarrow \overline{\mathbf{NE}(X)} = \mathbb{R}_{\geq 0} x_1 + \mathbb{R}_{\geq 0} x_2$.

Fact (A special case of Kovacs'94)

If $\rho = 2$, then $\overline{\mathbf{NE}(X)}$ is spanned by either:

1. (-2)-curve and elliptic curve,
2. two (-2)-curves,
3. two elliptic curves, or
4. two non-effective divisors x_1, x_2 with $x_i^2 = 0$.

Ex.

1. E is a line on S, $F := \mathbf{h} - E$. $F^2 = 0$ (elliptic). Then the ext. rays are spanned by E and F.
2. E_1 is a conic on S, $E_2 := \mathbf{h} - E_1$. $E_2^2 = -2$ (conic). Then the ext. rays are spanned by E_1, E_2.
3. F_1 is a complete intersection $(2) \cap (2) \subset \mathbb{P}^3$. $F_2 := 2\mathbf{h} - F_1$. $F_1^2 = F_2^2 = 0$ (two elliptics). Then the ext. rays are spanned by F_1, F_2.

Mori cone of smooth quartic surface ($\rho = 2$)

Lemma

Assume $^{\exists}E \simeq \mathbb{P}^1$ on a smooth quartic surface S and $\operatorname{Pic} S = \mathbb{Z}\mathrm{h} \oplus \mathbb{Z}E$. Let e be the degree of E.

1. If $e = 1$, then $\overline{\mathrm{NE}(S)}$ is spanned by E and elliptic curve $F = \mathrm{h} - E$.
2. if $e \geq 2$, then $\overline{\mathrm{NE}(S)}$ is spanned by E and E', where $E' \simeq \mathbb{P}^1$.

Proof.

Solve the Pell's equation

$$2x^2 + exy - y^2 = -1 \quad (\Longleftrightarrow \quad (x\mathrm{h} + yE)^2 = -2) \qquad \square$$

the classes of the other (-2)-curves

The classes of the other (-2)-curve E' is explicitly obtained as follows:

$e = d(E)$	the class of (-2)-curve E'
2	$\mathrm{h} - E$
3	$16h - 9E$
4	$2h - E$
5	$8h - 3E$
6	$3h - E$
7	$40h - 11E$
8	$4h - E$
9	$106000h - 23001E$
$\vdots$	$\vdots$

Theorem

Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing a line E. Suppose that $\operatorname{Pic} S = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E$. Let $C \subset S$ be a curve, let $F := \mathbf{h} - E$, and suppose that $D := C - 4\mathbf{h} \geq 0$.

Then

1. If $D \cdot E \geq -1$ and $D \neq nF$ for any $n \geq 2$, or $D = E$, then C is unobstructed.
2. If $D \cdot E = -2$ and $D \neq E$, then C is obstructed.

Theorem

Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing a rational curve $E \simeq \mathbb{P}^1$ of degree $e \geq 2$. Suppose that

$$\operatorname{Pic} S = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}E.$$

Let E' be another (-2)-curve on S, and let $C \subset S$ be a curve, and suppose that $D := C - 4\mathbf{h} \geq 0$.

1. If D is nef, $D = E$ or $D = E'$, then C is unobstructed.
2. If $D \cdot E = -2$ and $D \neq E$, then C is obstructed.

Thank you for your attention!

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